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Answers and Explanations to AB Practice Exam 1

ANSWERS AND EXPLANATIONS TO SECTION I

Problem 1. If $f(x)=5x^{\frac{4}{3}}$, then $f'(8)=$

We need to use Basic Differentiation to solve this problem.

Step 1: $f'(x)=\frac{4}{3}\left(5x^{\frac{1}{3}}\right)$

Step 2: Now all we have to do is Plug In 8 for x and simplify.

$$\frac{4}{3}\left(5\left(8^{\frac{1}{3}}\right)\right)=\frac{4}{3}(5(2))=\frac{40}{3}$$

The answer is (B).

Problem 2. $\lim\limits_{x\to\infty}\frac{5x^2-3x+1}{4x^2+2x+5}$ is

Step 1: To solve this problem, you need to remember how to evaluate Limits. Always do limit problems on the first pass. Whenever we have a limit of a polynomial fraction where $x\to\infty$, we divide the numerator and the denominator, separately, by the highest power of x in the fraction.

$$\lim_{x\to\infty}\frac{5x^2-3x+1}{4x^2+2x+5}=\lim_{x\to\infty}\frac{\frac{5x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{4x^2}{x^2}+\frac{2x}{x^2}+\frac{5}{x^2}}$$

Step 2: Simplify $\lim\limits_{x\to\infty}\frac{5-\frac{3}{x}+\frac{1}{x^2}}{4+\frac{2}{x}+\frac{5}{x^2}}$

Step 3: Now take the limit. Remember that the $\lim\limits_{x\to\infty}\frac{k}{x^n}=0$, if $n>0$, where k is a constant. Thus we get

$$\lim_{x\to\infty}\frac{5-\frac{3}{x}+\frac{1}{x^2}}{4+\frac{2}{x}+\frac{5}{x^2}}=\lim_{x\to\infty}\frac{5-0+0}{4+0+0}=\frac{5}{4}.$$

The answer is (D).

PROBLEM 3. If $f(x)=\dfrac{3x^2+x}{3x^2-x}$ then $f'(x)$ is

Step 1: We need to use the Quotient Rule to evaluate this derivative. Remember, the derivative of $\dfrac{u}{v}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$. But before we take the derivative, we should factor an x out of the top and bottom and cancel, simplifying the quotient.

$$f(x)=\frac{3x^2+x}{3x^2-x}=\frac{x(3x+1)}{x(3x-1)}=\frac{3x+1}{3x-1}$$

Step 2: Now take the derivative.

$$f'(x)=\frac{(3x-1)(3)-(3x+1)(3)}{(3x-1)^2}$$

Step 3: Simplify:

$$\frac{9x-3-9x-3}{(3x-1)^2}=\frac{-6}{(3x-1)^2}$$

The answer is (C).

PROBLEM 4. If the function f is continuous for all real numbers and if $f(x)=\dfrac{x^2-7x+12}{x-4}$ when $x\neq 4$, then $f(4)=$

This problem is testing your knowledge of Continuity.

Step 1: Notice that if we plug 4 into the numerator and denominator we get $\dfrac{0}{0}$, which is undefined. So, the first thing that we should do is factor the numerator. What we are looking for is a common factor in the numerator and denominator. If we find a common factor, we can cancel the factors and simplify the problem.

We get $f(x)=\dfrac{x^2-7x+12}{x-4}=\dfrac{(x-3)(x-4)}{x-4}=(x-3)$.

Step 2: Now we Plug In 4 for x and we get 1.

The answer is (A).

Problem 5. If $x^2 - 2xy + 3y^2 = 8$, then $\frac{dy}{dx} =$

Whenever we have a polynomial where the x's and y's are not separated we need to use Implicit Differentiation to find the derivative.

Step 1: Take the derivative of everything with respect to x.

$$2x\frac{dx}{dx} - 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 6y\frac{dy}{dx} = 0 \quad \text{Remember that } \frac{dx}{dx} = 1!$$

Step 2: Simplify and then put all of the terms containing $\frac{dy}{dx}$ on one side, and all of the other terms on the other side.

$$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$$

$$-2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 2y - 2x$$

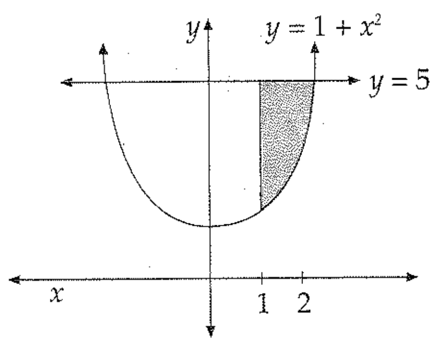
Factor out the $\frac{dy}{dx}$, then isolate it.

$$\frac{dy}{dx}(6y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{2y - 2x}{(6y - 2x)} = \frac{y - x}{3y - x}$$

The answer is (E).

Problem 6.



Which of the following integrals correctly corresponds to the area of the region in the figure above between the curve $y = 1 + x^2$ and the line $y = 5$ from $x = 1$ to $x = 2$?

We use integrals to find the Area Between Two Curves. If the top curve of a region is $f(x)$ and the bottom curve of a region is $g(x)$, from $x = a$ to $x = b$, then the area is found by the integral

$$\int_a^b [f(x) - g(x)]\,dx$$

Step 1: The top curve here is the line $y = 5$, and the bottom curve is $y = 1 + x^2$, and the region extends from the line $x = 1$ to the line $x = 2$. Thus, the integral for the area is

$$\int_1^2 \left[(5) - \left(1 + x^2\right)\right]dx = \int_1^2 \left(4 - x^2\right)dx$$

The answer is (B).

Problem 7. If $f(x) = \sec x + \csc x$, then $f'(x) =$

This question is testing whether you know your Derivatives of Trigonometric Functions. If you do, this is an easy problem.

Step 1: The derivative of $\sec x$ is $\sec x \tan x$ and the derivative of

$\csc x$ is $-\csc x \cot x$. That makes the derivative here $\sec x \tan x - \csc x \cot x$.

The answer is (E).

Problem 8. An equation of the line normal to the graph of $y = \sqrt{\left(3x^2 + 2x\right)}$ at $(2, 4)$ is

Here we do everything that we normally do for finding the Equations of Tangent Lines, except that we use the negative reciprocal of the slope to find the normal line. This is because the normal line is perpendicular to the tangent line.

Step 1: First, find the slope of the tangent line.

$$\frac{dy}{dx} = \frac{1}{2}\left(3x^2 + 2x\right)^{-\frac{1}{2}}(6x + 2)$$

Step 2: DON'T SIMPLIFY. Immediately Plug In $x = 2$. We get:

$$\frac{dy}{dx} = \frac{1}{2}\left(3x^2 + 2x\right)^{-\frac{1}{2}}(6x + 2) = \frac{1}{2}\left(3(2)^2 + 2(2)\right)^{-\frac{1}{2}}(6(2) + 2) = \frac{1}{2}(16)^{-\frac{1}{2}}(14) = \frac{7}{4}$$

This means that the slope of the tangent line at $x = 2$ is $\frac{7}{4}$, so the slope of the normal line is $-\frac{4}{7}$.

Step 3: Then the equation of the tangent line is $(y - 4) = -\frac{4}{7}(x - 2)$.

Step 4: Multiply through by 7 and simplify.

$$7y - 28 = -4x + 8$$
$$4x + 7y = 36$$

The answer is (E).

Problem 9. $\int_{-1}^{1} \frac{4}{1+x^2}\, dx =$

You should recognize this integral as one of the Inverse Trigonometric Integrals.

Step 1: As you should recall, $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$. The 4 is no big deal, just multiply the integral by 4 to get $4\tan^{-1}(x)$. Then we just have to evaluate the limits of integration.

Step 2: $4\tan^{-1}(x)\Big|_{-1}^{1} = 4\tan^{-1}(1) - 4\tan^{-1}(-1) = 4\left(\frac{\pi}{4}\right) - 4\left(-\frac{\pi}{4}\right) = 2\pi$

The answer is (D).

Problem 10. If $f(x) = \cos^2 x$, then $f''(\pi) =$

This problem is just asking us to find a higher order Derivative of a Trigonometric Function.

Step 1: The first derivative requires the chain rule:

$$f(x) = \cos^2 x$$
$$f'(x) = 2(\cos x)(-\sin x) = -2\cos x \sin x$$

Step 2: The second derivative requires the product rule:

$$f'(x) = -2\cos x \sin x$$
$$f''(x) = -2(\cos x \cos x - \sin x \sin x) = -2\left(\cos^2 x - \sin^2 x\right)$$

Step 3: Now we Plug In π for x and simplify.

$$-2\left(\cos^2(\pi) - \sin^2(\pi)\right) = -2(1 - 0) = -2$$

The answer is (A).

Problem 11. If $f(x)=\frac{5}{x^2+1}$ and $g(x)=3x$ then $g(f(2))=$

Step 1: To find $g(f(x))$, all you need to do is to replace all of the x's in $g(x)$ with $f(x)$'s.

$$g(f(x))=3f(x)=3\left(\frac{5}{x^2+1}\right)=\frac{15}{x^2+1}$$

Step 2: Now all we have to do is Plug In 2 for x.

$$g(f(2))=\frac{15}{2^2+1}=3$$

The answer is (C).

Problem 12. $\int x\sqrt{5x^2-4}\,dx=$

Any time we have an integral with an x factor whose power is one less than another x factor, we can try to do the integral with u-substitution. This is our favorite technique for doing integration and the most important one to master.

Step 1: Let $u = 5x^2 - 4$ and $du = 10xdx$ and so $\frac{1}{10}du = xdx$.

Then we can rewrite the integral as:

$$\int x\sqrt{5x^2-4}\,dx=\frac{1}{10}\int u^{\frac{1}{2}}du.$$

Step 2: Now this becomes a basic integral.

$$\frac{1}{10}\int u^{\frac{1}{2}}du=\frac{1}{10}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)+C=\frac{1}{15}u^{\frac{3}{2}}+C$$

Step 3: Reverse the substitution and we get: $\frac{1}{15}\left(5x^2-4\right)^{\frac{3}{2}}+C$

The answer is (B).

Problem 13. The slope of the line tangent to the graph of $3x^2 + 5\ln y = 12$ at $(2,1)$ is

This is another Equation of a Tangent Line problem, combined with Implicit Differentiation. Often the AP Examination has more than one tangent line problem, so make sure that you can do these well!

By the way, do you remember the derivative of $\ln(f(x))$? It is $\frac{f'(x)}{f(x)}$.

Step 1: First, we take the derivative of the equation.

$$6x\frac{dx}{dx} + \frac{5}{y}\frac{dy}{dx} = 0$$

Step 2: Next, we simplify and solve for $\frac{dy}{dx}$.

$$6x + \frac{5}{y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-6xy}{5}$$

Step 3: Now we Plug In 2 for x and 1 for y to get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{-6(2)(1)}{5} = \frac{-12}{5}$$

The answer is (A).

Note: We could have plugged in directly for x and y after simplifying. On a more complicated derivative you ALWAYS want to Plug In right after you differentiate. On a simple one such as this, the choice is up to you.

Problem 14. The equation $y = 2 - 3\sin\frac{\pi}{4}(x-1)$ has a fundamental period of

The AP people expect you to remember a lot of your trigonometry, so if you're rusty, review the unit in the Appendix.

Step 1: In an equation of the form $f(x) = A \sin B(x \pm C) \pm D$, you should know four components. The amplitude of the equation is A, the horizontal or phase shift is $\pm C$, the vertical shift is $\pm D$, and the fundamental period is $\frac{2\pi}{B}$.

The same is true for $f(x) = A \cos B(x \pm C) \pm D$.

Step 2: All we have to do is Plug Into the formula for the period.

$$\frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 8$$

The answer is (D).

Problem 15. If $f(x) = \begin{cases} x^2 + 5 & \text{if } x < 2 \\ 7x - 5 & \text{if } x \geq 2 \end{cases}$, for all real numbers x, which of the following must be true?

I. $f(x)$ is continuous everywhere.

II. $f(x)$ is differentiable everywhere.

III. $f(x)$ has a local minimum at $x = 2$.

This problem is testing your knowledge of the rules of continuity and differentiability. While the more formal treatment is located in the unit on Continuity, here we'll go directly to a shortcut to the right answer. This type of function is called a piecewise function because it is broken into two or more pieces, depending on the value of x that one is looking at.

Step 1: If a piecewise function is continuous at a point a, then when you plug a into each of the pieces of the function, you should get the same answer. The function consists of a pair of polynomials (Remember that all polynomials are continuous!), where the only point that might be a problem is $x = 2$. So here we'll plug 2 into both pieces of the function to see if we get the same value. If we do, then the function is continuous. If we don't, then it's discontinuous. At $x = 2$, the upper piece is equal to 9 and the lower piece is also equal to 9. So the function is continuous everywhere, and I is true. You should then eliminate answer choice C.

Step 2: If a piecewise function is differentiable at a point a, then when you plug a into each of the derivatives of the pieces of the function, you should get the same answer. It is the same idea as in Step 1. So here we will plug 2 into the derivatives of both pieces of the function to see if we get the same value. If we do, then the function is differentiable. If we don't, then it is non-differentiable at $x = 2$.

The derivative of the upper piece is $2x$, and at $x = 2$, the derivative is 4.

The derivative of the lower piece is 7 everywhere.

Because the two derivatives are not equal, the function is not differentiable everywhere and II is false. You should then eliminate answer choices B and E.

Step 3: The slope of the function to the left of $x = 2$ is 4. The slope of the function to the right of $x = 2$ is 7. If the slope of a function has the same sign on either side of a point then the function cannot have a local minimum or maximum at that point. So **III** is false because of what we found in Step 2.

The answer is (A).

PROBLEM 16. For what value of x does the function $f(x) = x^3 - 9x^2 - 120x + 6$ have a local minimum?

This problem requires you to know how to find Maxima/Minima. This is a part of curve sketching and is one of the most important parts of Differential Calculus. A function has *critical points* where the derivative is zero or undefined (which is never a problem when the function is an ordinary polynomial). After finding the critical points we test them to determine whether they are maxima or minima or something else.

Step 1: First, as usual, take the derivative and set it equal to zero.

$$f'(x) = 3x^2 - 18x - 120$$
$$3x^2 - 18x - 120 = 0$$

Step 2: Find the values of x that make the derivative equal to zero. These are the critical points.

$$3x^2 - 18x - 120 = 0$$
$$x^2 - 6x - 40 = 0$$
$$(x-10)(x+4) = 0$$
$$x = \{10, -4\}$$

Step 3: In order to determine whether a critical point is a maximum or a minimum, we need to take the second derivative. $f''(x) = 6x - 18$

Step 4: Now we plug the critical points from Step 2 into the second derivative. If it yields a negative value, then the point is a maximum. If it yields a positive value, then the point is a minimum. If it yields zero, it is neither, and is most likely a point of inflection.

$$6(10) - 18 = 42$$
$$6(-4) - 18 = -42$$

Therefore 10 is a minimum.

The answer is (A).

PROBLEM 17. The acceleration of a particle moving along the x-axis at time t is given by $a(t) = 4t - 12$. If the velocity is 10 when $t = 0$ and the position is 4 when $t = 0$, then the particle is changing direction at

Step 1: Because acceleration is the derivative of velocity, if we know the acceleration of a particle, we can find the velocity by integrating the acceleration with respect to t.

$$\int (4t - 12)dt = 2t^2 - 12t + C$$

Next, because the velocity is 10 at $t = 0$, we can Plug In 0 for t and solve for the constant.

$$2(0)^2 - 12(0) + C = 10.$$

Therefore $C = 10$ and the velocity, $v(t)$, is $2t^2 - 12t + 10$

Step 2: In order to find when the particle is changing direction we need to know when the velocity is equal to zero, so we set $v(t) = 0$ and solve for t.

$$\begin{aligned} 2t^2 - 12t + 10 &= 0 \\ t^2 - 6t + 5 &= 0 \\ (t-5)(t-1) &= 0 \\ t &= \{1, 5\} \end{aligned}$$

Now, provided that the acceleration is not also zero at $t = \{1, 5\}$, the particle will be changing direction at those times. The acceleration is found by differentiating the equation for velocity with respect to time: $a(t) = 4t - 12$. This is not zero at either $t = 1$ or $t = 5$. Therefore, the particle is changing direction when $t = 1$ and $t = 5$.

The answer is (D).

PROBLEM 18. The average value of the function $f(x) = (x-1)^2$ on the interval from $x = 1$ to $x = 5$ is

Step 1: If you want to find the average value of $f(x)$ on an interval $[a, b,]$ you need to evaluate the integral $\frac{1}{b-a}\int_a^b f(x)\,dx$.

So here we would evaluate the integral $\frac{1}{5-1}\int_1^5 (x-1)^2\,dx$.

Step 2: $\frac{1}{5-1}\int_1^5 (x-1)^2\,dx = \frac{1}{4}\int_1^5 (x^2 - 2x + 1)\,dx$

$$= \frac{1}{4}\left(\frac{x^3}{3} - x^2 + x\right)\Bigg|_1^5 = \frac{1}{4}\left[\left(\frac{5^3}{3} - 5^2 + 5\right) - \left(\frac{1}{3} - 1 + 1\right)\right]$$

$$= \frac{1}{4}\left(\frac{125}{3} - 20 - \frac{1}{3}\right) = \frac{64}{12} = \frac{16}{3}$$

The answer is (B).

PROBLEM 19. $\int\left(e^{3\ln x}+e^{3x}\right)dx=$

This problem requires that you know your rules of Exponential Functions.

Step 1: First of all, $e^{3\ln x} = e^{\ln x^3} = x^3$. So we can rewrite the integral as

$$\int\left(e^{3\ln x}+e^{3x}\right)dx=\int\left(x^3+e^{3x}\right)dx.$$

Step 2: The rule for the integral of an exponential function is $\int e^k dx=\frac{1}{k}e^{kx}+C$.

Now we can do this integral. $\int\left(x^3+e^{3x}\right)dx=\frac{x^4}{4}+\frac{1}{3}e^{3x}+C$.

The answer is (E).

PROBLEM 20. If $f(x)=\sqrt{\left(x^3+5x+121\right)}\left(x^2+x+11\right)$ then $f'(0)=$

This problem is just a complicated derivative, requiring you to be familiar with the Chain Rule and the Product Rule.

Step 1: $f'(x)=\frac{1}{2}\left(x^3+5x+121\right)^{-\frac{1}{2}}\left(3x^2+5\right)\left(x^2+x+11\right)+\left(x^3+5x+121\right)^{\frac{1}{2}}(2x+1)$

Step 2: Whenever a problem asks you to find the value of a complicated derivative at a particular point, NEVER simplify the derivative. Immediately Plug In the value for x and do arithmetic instead of algebra.

$$f'(x)=\frac{1}{2}\left(0^3+5(0)+121\right)^{-\frac{1}{2}}\left(3(0)^2+5\right)\left(0^2+(0)+11\right)+\left((0)^3+5(0)+121\right)^{\frac{1}{2}}(2(0)+1)$$
$$=\frac{1}{2}(121)^{-\frac{1}{2}}(5)(11)+(121)^{\frac{1}{2}}(1)=\frac{5}{2}+11=\frac{27}{2}$$

The answer is (B).

PROBLEM 21. If $f(x)=5^{3x}$ then $f'(x)=$

This problem requires you to know how to find the Derivative of an Exponential Function. The rule is: If a function is of the form $a^{f(x)}$, its derivative is $a^{f(x)}(\ln a)\ f'(x)$. Now all we have to do is follow the rule!

Step 1: $f(x)=5^{3x}$

$f'(x)=5^{3x}(\ln 5)(3)$

Step 2: If you remember your rules of logarithms, $3\ln 5 = \ln(5^3) = \ln 125$.

So we can rewrite the answer to $f'(x) = 5^{3x}(\ln 5)(3) = 5^{3x}\ln 125$.

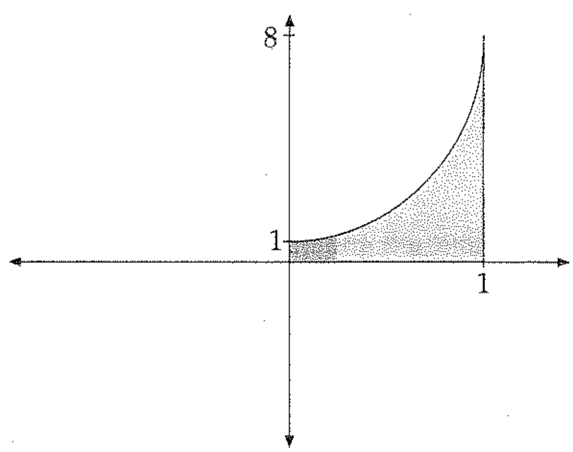
The answer is (A).

Problem 22. A solid is generated when the region in the first quadrant enclosed by the graph of $y = (x^2 + 1)^3$, the line $x = 1$, and the x-axis, is revolved about the x-axis. Its volume is found by evaluating which of the following integrals?

This problem requires you to know how to find the Volume of a Solid of Revolution.

If you have a region between two curves, from $x = a$ to $x = b$, then the volume generated when the region is revolved around the x-axis is: $\pi\int_a^b [f(x)]^2 - [g(x)]^2\, dx$, if $f(x)$ is above $g(x)$ throughout the region.

Step 1: First, we have to determine what the region looks like. The curve looks like this:



The shaded region is the part that we are interested in. Notice that the curve is always above the x-axis (which is $g(x)$). Now we just follow the formula:

$$\pi\int_0^1 \left[\left(x^2+1\right)^3\right]^2 - [0]^2\, dx = \pi\int_0^1 \left(x^2+1\right)^6 dx$$

The answer is (D).

PROBLEM 23. $\lim_{x \to 0} 4\frac{\sin x \cos x - \sin x}{x^2} =$

This problem requires us to evaluate the Limit of a Trigonometric Function.

There are two important trigonometric limits to memorize:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Step 1: The first step that we always take when evaluating the limit of a trigonometric function is to rearrange the function so that it looks like some combination of the limits above. We can do this by factoring a sin x out of the numerator.

Now we can break this into limits that we can easily evaluate.

$$\lim_{x \to 0} 4\frac{\sin x \cos x - \sin x}{x^2} = 4\lim_{x \to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\cos x - 1}{x}\right)$$

$$\left(\text{Note that } \lim_{x \to 0} \frac{1 - \cos x}{x} = -\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.\right)$$

Step 2: Now if we take the limit as $x \to 0$ we get: $4(1)(0) = 0$

The answer is (D).

PROBLEM 24. If $\frac{dy}{dx} = \frac{(3x^2 + 2)}{y}$ and $y = 4$ when $x = 2$, then when $x = 3$, y

This is a very basic Differential Equation. As with many of the more difficult topics in Calculus, the AP examination only tends to ask us to solve very straightforward differential equations. In fact, on the AB examination, you are only going to need to know one technique for getting these right. It is called *Separation of Variables*.

Step 1: First, take all of the terms with a y in them and put them on the left side of the equal sign. Take all of the terms with an x in them and put them on the right side of the equal sign. Then we get:

$$ydy = (3x^2 + 2)dx$$

Step 2: Now integrate both sides.

$$\int ydy = \int (3x^2 + 2)dx$$

$$\frac{y^2}{2} = x^3 + 2x + c$$

Notice how we only use one constant. All we have to do now is solve for C. We do this by Plugging In 2 for x and 4 for y.

$$\frac{16}{2} = 2^3 + 4 + C$$
$$C = -4$$

So we can rewrite the equation as $\frac{y^2}{2} = x^3 + 2x - 4$.

Step 3: Now if we Plug In 3 for x, we will get y.

$$\frac{y^2}{2} = 27 + 6 - 4$$
$$y^2 = 58$$
$$y = \pm\sqrt{58}$$

The answer is (E).

Problem 25. $\int \frac{dx}{9+x^2} =$

This is another Inverse Trigonometric Integral.

Step 1: We know that $\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C$.

(See problem 9 if you're not sure of this.) The trick here is to get the denominator of the fraction in the integrand to be of the correct form. If we factor 9 out of the denominator we get:

$$\int \frac{dx}{9+x^2} = \int \frac{dx}{9\left(1+\frac{x^2}{9}\right)} = \frac{1}{9}\int \frac{dx}{1+\frac{x^2}{9}} = \frac{1}{9}\int \frac{dx}{1+\left(\frac{x}{3}\right)^2}$$

Step 2: Now if we use u-substitution we will be able to evaluate this integral.

Let $u = \frac{x}{3}$ and $du = \frac{1}{3}dx$ or $3du = dx$. Then we have:

$$\frac{1}{9}\int \frac{dx}{1+\left(\frac{x}{3}\right)^2} = \frac{1}{9}\int \frac{3du}{1+u^2} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}(u) + C.$$

Step 3: Now all we have to do is reverse the u-substitution and we're done.

$$\frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

The answer is (B).

PROBLEM 26. If $f(x)=\cos^3(x+1)$ then $f'(\pi)=$

Think of $\cos^3(x+1)$ as $[\cos(x+1)]^3$.

Step 1: First, we take the derivative of the outside function and ignore the inside functions. The derivative of u^3 is $3u^2$

We get: $\frac{d}{dx}[u]^3=3[u]^2$.

Step 2: Next, we take the derivative of the cosine term and multiply. The derivative of $\cos u$ is $-\sin u$.

$$\frac{d}{dx}[\cos(u)]^3=-3[\cos(u)]^2\sin(u)$$

Step 3: Finally, we take the derivative of $x+1$ and multiply. The derivative of $x+1$ is 1.

$$\frac{d}{dx}[\cos(x+1)]^3=-3[\cos(x+1)]^2\sin(x+1)$$

The answer is (A).

Note: If you had trouble with this problem, you should review the section on **The Chain Rule.**

PROBLEM 27. $\int x\sqrt{x+3}\,dx=$

We can do this integral with u-substitution.

Step 1: Let $u=x+3$. Then $du=dx$ and $u-3=x$.

Step 2: Substituting, we get:

$$\int x\sqrt{x+3}\,dx=\int(u-3)u^{\frac{1}{2}}\,du$$

Why is this better than the original integral, you might ask? Because now we can distribute and the integral becomes easy.

Step 3: When we distribute, we get:

$$\int(u-3)u^{\frac{1}{2}}\,du=\int\left(u^{\frac{3}{2}}-3u^{\frac{1}{2}}\right)du$$

Step 4: Now we can integrate:

$$\int\left(u^{\frac{3}{2}}-3u^{\frac{1}{2}}\right)du=\frac{2}{5}u^{\frac{5}{2}}-3\cdot\frac{2}{3}u^{\frac{3}{2}}+C$$

Step 5: Substituting back, we get:

$$\frac{2}{5}u^{\frac{5}{2}}-3\cdot\frac{2}{3}u^{\frac{3}{2}}+C=\frac{2}{5}(x+3)^{\frac{5}{2}}-2(x+3)^{\frac{3}{2}}+C$$

The answer is (C).

Problem 28. If $f(x)=\ln(\ln(1-x))$, then $f'(x)=$

Here, we use the chain rule.

Step 1: First, take the derivative of the outside function.

The derivative of $\ln u$ is $\frac{du}{u}$.

We get:

$$\frac{d}{dx}\ln(\ln(\ ))=\frac{1}{\ln(\ )}$$

Step 2: Now we take the derivative of the function in the denominator. Once again, the function is $\ln u$.

We get:

$$\frac{d}{dx}\ln(\ln(1-x))=\frac{1}{\ln(1-x)}\cdot\frac{-1}{1-x}=-\frac{1}{(1-x)\ln(1-x)}.$$

The answer is (D).

Problem 29. $\int_0^{\frac{\pi}{4}}\sin x\ dx+\int_{\frac{-\pi}{4}}^{0}\cos x\ dx=$

This is a pair of basic Trigonometric Integrals. You should have memorized several trigonometric integrals, particularly $\int\sin x\ dx=-\cos x+C$ and $\int\cos x\ dx=\sin x+C$

Step 1: $\int_0^{\frac{\pi}{4}}\sin x\ dx+\int_{\frac{-\pi}{4}}^{0}\cos x\ dx=-\cos x\Big|_0^{\frac{\pi}{4}}+\sin x\Big|_{\frac{-\pi}{4}}^{0}$

Step 2: Now we evaluate the limits of integration, and we're done.

$$-\cos x\Big|_0^{\frac{\pi}{4}}+\sin x\Big|_{-\frac{\pi}{4}}^{0}=\left[\left(-\cos\frac{\pi}{4}\right)-\left(-\cos(0)\right)\right]+\left[\left(\sin(0)\right)-\left(\sin\left(-\frac{\pi}{4}\right)\right)\right]=-\frac{1}{\sqrt{2}}+1+0+\frac{1}{\sqrt{2}}=1$$

The answer is (D).

PROBLEM 30. Boats A and B leave the same place at the same time. Boat A heads due north at 12 km/hr. Boat B heads due east at 18 km/hr. After 2.5 hours, how fast is the distance between the boats increasing (in km/hr)?

Step 1: The boats are moving at right angles to each other and are thus forming a right triangle with the distance between them forming the hypotenuse. Whenever we see right triangles in related rates problems, we look to use the Pythagorean Theorem. Call the distance that Boat A travels y, and the distance that Boat B travels x. Then the rate at which Boat A goes north is $\frac{dy}{dt}$, and the rate at which Boat B travels is $\frac{dx}{dt}$. The distance between the two boats is z, and we are looking for how fast z is growing, which is $\frac{dz}{dt}$. Now we can use the Pythagorean Theorem to set up the relationship: $x^2 + y^2 = z^2$

Step 2: Differentiating both sides we obtain:

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt} \text{ or } x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}.$$

Step 3: After 2.5 hours, Boat A has traveled 30 km and Boat B has traveled 45 km. Because of the Pythagorean Theorem, we also know that, when $y = 30$ and $x = 45$, $z = 54.08$.

Step 4: Now we plug everything into the equation from Step 2 and solve for $\frac{dz}{dt}$:

$$(45)(18)+(30)(12)=(54.08)\frac{dz}{dt}$$

$$1170=(54.08)\frac{dz}{dt}$$

$$21.63=\frac{dz}{dt}$$

The answer is (A).

PROBLEM 31. $\lim_{h\to 0}\dfrac{\tan\left(\frac{\pi}{6}+h\right)-\tan\left(\frac{\pi}{6}\right)}{h}=$

This may *appear* to be a limit problem, but it is *actually* testing to see whether you know The Definition of the Derivative.

Step 1: You should recall that the Definition of the Derivative says

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x).$$

Thus, if we replace $f(x)$ with tan (x), we can rewrite the problem as:

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}=\left[\tan(x)\right]'.$$

Step 2: The derivative of tan x is $\sec^2 x$. Thus

$$\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{6}+h\right)-\tan\left(\frac{\pi}{6}\right)}{h}=\sec^2\left(\frac{\pi}{6}\right).$$

Step 3: Because $\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}, \sec^2\left(\frac{\pi}{6}\right)=\frac{4}{3}$.

The answer is (B).

> **Note:** If you had trouble with this problem, you should review the units on The Definition of the Derivative and Derivatives of Trigonometric Functions.

PROBLEM 32. If $\int_{30}^{100} f(x)dx = A$ and $\int_{50}^{100} f(x)dx = B$ then $\int_{30}^{50} f(x)dx =$

This question is testing your knowledge of the rules of Definite Integrals.

Step 1: Generally speaking, $\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$.

So here, $\int_{30}^{50} f(x)dx+\int_{50}^{100} f(x)dx=\int_{30}^{100} f(x)dx$.

If we substitute $\int_{30}^{100} f(x)dx=A$ and $\int_{50}^{100} f(x)dx=B$, we get $\int_{30}^{50} f(x)dx+B=A$.

The answer is (B).

Problem 33. If $f(x) = 3x^2 - x$, and $g(x) = f^{-1}(x)$, then $g'(10)$ could be

This problem requires you to know how to find the Derivative of an Inverse Function.

Step 1: The rule for finding the derivative of an inverse function is:

$$\text{If } y = f(x) \text{ and if } g(x) = f^{-1}(x) \text{ then } g'(x) = \frac{1}{f'(y)}.$$

Step 2: In order to use the formula, we need to find the derivative of f and the value of x that corresponds to $y = 10$.

First, $f'(x) = 6x - 1$. Second, when $y = 10$ we get $10 = 3x^2 - x$.

If we solve this for x we get $x = 2$ (and $x = -\frac{5}{3}$ but we'll use 2. It's easier.)

Step 3: Plugging into the formula, we get $\frac{1}{f'(y)} = \frac{1}{(6)(2)-1} = \frac{1}{11}$.

The answer is (E).

> **Note:** There was another possible answer using $x = -\frac{5}{3}$, but that doesn't give us one of the answer choices. Generally, the AP examination sticks to the easier answer. They are testing whether you know what to do and usually NOT trying to trick you.

Problem 34. The graph of $y = x^3 - 5x^2 + 4x + 2$ has a local minimum at

This is another Maxima/Minima question.

Step 1: Take the derivative of the function and set it equal to zero.

$$f'(x) = 3x^2 - 10x + 4 = 0$$

Step 2: Use the quadratic formula to solve for x. You should get $x = \{2.87, 0.46\}$.

Step 3: Now take the second derivative of the function.

$$f''(x) = 6x - 10$$

Step 4: Plug each of the critical values from Step 2 into the second derivative. If you get a positive value, the point is a minimum. If you get a negative value, the point is a maximum. If you get zero, the point is probably a point of inflection (don't worry about that here).

$$f''(2.87) = 7.21$$
$$f''(.46) = -7.21$$

So 2.87 is the x-coordinate of the minimum. To find the y-coordinate, just plug 2.87 into $f(x)$ and you get -4.06.

The answer is (C).

Problem 35. The volume generated by revolving about the y-axis the region enclosed by the graphs $y = 9 - x^2$ and $y = 9 - 3x$, for $0 \le x \le 2$, is

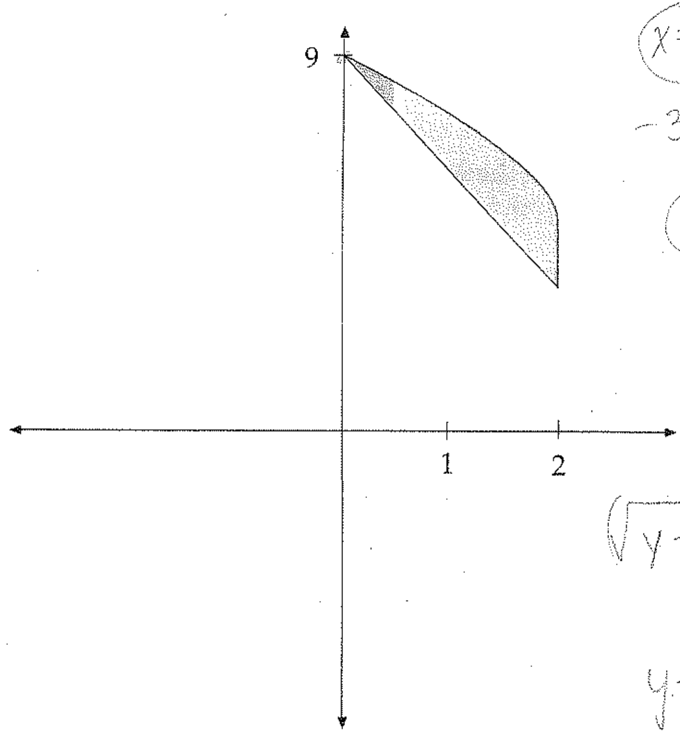
This is another Volume of a Solid of Revolution problem. As you should have noticed by now, these are very popular on the AP Examination and show up in both the multiple-choice section and in the Long Problem section. If you are not good at these, go back and review the unit carefully. You cannot afford to get these wrong on the AP! The good thing about *this* volume problem is that it is in the calculator part of the multiple choice section, so you can use a graphing calculator to assist you.

Step 1: First, graph the two curves on the same set of axes. The graph should look like this:

$x = \sqrt{y-9}$

$-3x = y - 9$

$x = -\frac{1}{3}y + 3$



$(\sqrt{y-9})^2 = (-\frac{1}{3}y + 3)^2$

$y - 9 = (-\frac{1}{3}y + 3)(-\frac{1}{3}y + 3)$

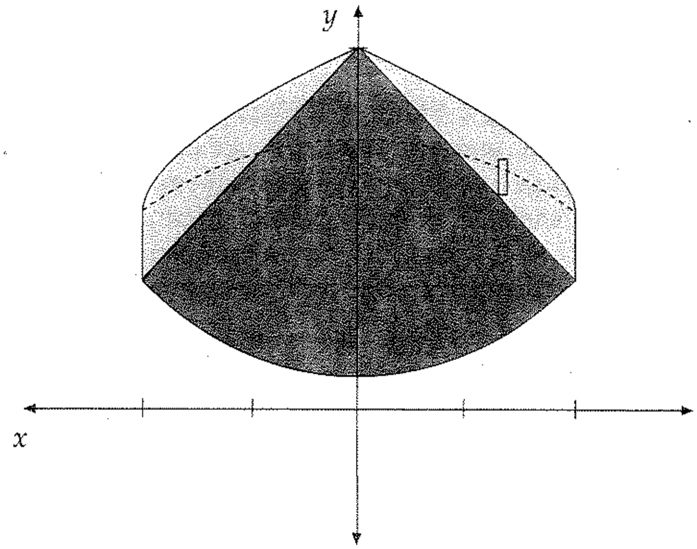
$y - 9 = \frac{1}{9}y^2 - y - y + 9$

$y - 9 = \frac{1}{9}y^2 - 2y + 9$

$\pi \int_3^9 (\frac{1}{9}y^2 - 3y + 18)\,dy$

$0 = \frac{1}{9}y^2 - 3y + 18$

Step 2: We are being asked to rotate this region around the y-axis, and both of the functions are in terms of x, so we should use the method of shells. We use this method whenever we take a vertical slice of a region and rotate it around an axis parallel to the slice (review the unit if you are not sure what it means). This will give us a region that looks like this:



Step 3: The formula for the method of shells says that if you have a region between two curves, $f(x)$ and $g(x)$ from $x = a$ to $x = b$, then the volume generated when the region is revolved around the y-axis is: $2\pi\int_a^b x[f(x)-g(x)]dx$; if $f(x)$ is above $g(x)$ throughout the region.Thus our integral is.

$$2\pi\int_0^2 x[(9-x^2)-(9-3x)]dx$$

We can simplify this integral to $2\pi\int_0^2 x(3x-x^2)dx = 2\pi\int_0^2(3x^2-x^3)dx$.

Step 4: Evaluate the integral:

$$2\pi\int_0^2(3x^2-x^3)dx = 2\pi\left(x^3-\frac{x^4}{4}\right)\Bigg|_0^2 = 8\pi$$

The answer is (C).

Problem 36. The average value of the function $f(x) = \ln^2 x$ on the interval [2,4] is

This problem requires you to be familiar with the Mean Value Theorem for Integrals which we use to find the average value of a function.

Step 1: If you want to find the average value of $f(x)$ on an interval $[a, b]$, you need to evaluate the integral. $\frac{1}{b-a}\int_a^b f(x)dx$ So here we evaluate the integral $\frac{1}{2}\int_2^4 \ln^2 x\ dx$.

You have to do this integral on your calculator because you do not know how to evaluate this integral analytically unless you are very good with integration by parts!

Use **fnint**. Divide this by 2 and you will get 1.204.

The answer is (B).

Problem 37. $\frac{d}{dx}\int_0^{3x}\cos(t)dt =$

This problem is testing your knowledge of the Second Fundamental Theorem of Calculus. The theorem states that $\frac{d}{dx}\int_a^u f(t)dt = f(u)\frac{du}{dx}$, where a is a constant and u is a function of x. So all we have to do is follow the theorem. $\frac{d}{dx}\int_0^{3x}\cos(t)dt = 3\cos 3x$.

The answer is (E).

Problem 38. If the definite integral $\int_1^3 (x^2+1)dx$ is approximated by using the Trapezoid Rule with $n = 4$, the error is

This problem will require you to be familiar with the Trapezoid Rule. This is very easy to do on the calculator, and some of you may even have written programs to evaluate this. Even if you haven't, the formula is easy. The area under a curve from $x = a$ to $x = b$, divided into n intervals is approximated by the Trapezoid Rule and is

$$\left(\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\left[y_0 + 2y_1 + 2y_2 + 2y_3 \ldots + 2y_{n-2} + 2y_{n-1} + y_n\right]$$

This formula may look scary, but it actually is quite simple, and the AP Examination never uses a very large value for n anyway.

Step 1: $\frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$. Plugging into the formula, we get:

$$\frac{1}{4}\left[\left(1^2+1\right)+2\left(1.5^2+1\right)+2\left(2^2+1\right)+2\left(2.5^2+1\right)+\left(3^2+1\right)\right].$$

This is easy to Plug Into your calculator and you will get 10.75 or $\frac{43}{4}$.

Step 2: In order to find the error, we now need to know the actual value of the integral.

$$\int_1^3 x^2 + 1dx = \frac{x^3}{3} + x\Big|_1^3 = \frac{32}{3} \text{ or } 10.666.$$

Step 3: The error is $\frac{43}{4}-\frac{32}{3}=\frac{1}{12}$.

The answer is (C).

PROBLEM 39. The radius of a sphere is increasing at a rate proportional to itself. If the radius is 4 initially, and the radius is 10 after two seconds, then what will the radius be after three seconds?

This is not a Related Rate problem, this is a Differential Equation! It just happens to involve a rate.

Step 1: If we translate the first sentence into an equation we get: $\frac{dR}{dt}=kR$.

Put all of the terms that contain an R on the left of the equals sign, and all of the terms that contain a t on the right hand side. $\frac{dR}{R}=kdt$

Step 2: Integrate both sides: $\int\frac{dR}{R}=k\int dt$.

Step 3: If we solve this for R we get $R=Ce^{kt}$ (see the Unit on Differential Equations).

Now we need to solve for C and k. First we solve for C by Plugging In the information that the radius is 4 initially. This means that $R=4$ when $t=0$.

$$4=Ce^0 \text{ then } C=4.$$

Next we solve for k by Plugging In the information that $R=10$ when $t=2$.

$$10=4e^{2k}$$
$$\frac{5}{2}=e^{2k}$$
$$\ln\frac{5}{2}=2k$$
$$\frac{1}{2}\ln\frac{5}{2}=k$$

Step 4: Now we have our final equation: $R=4e^{\left(\frac{1}{2}\ln\frac{5}{2}\right)t}$.

If we Plug In $t=3$ we get: $R=4e^{\left(\frac{1}{2}\ln\frac{5}{2}\right)(3)}=15.811$

The answer is (C).

Problem 40. Use differentials to approximate the change in the volume of a sphere when the radius is increased from 10 to 10.02 cm.

The volume of a sphere is $V = \frac{4}{3}\pi R^3$. Using differentials, the change will be: $dV = 4\pi R^2 dR$.

Substitute in $R = 10$ and $dR = .02$, and we get:

$$dV = 4\pi(10^2)(.02)$$

$$dV = 8\pi \approx 25.133 cm^3$$

The answer is (E).

Problem 41. $\int \ln 2x \ dx =$

This is a simple integral that we do using Integration By Parts. The AB Examination only has the simplest of these types of integrals, although the BC Examination has harder ones. Furthermore, you should memorize that $\int \ln(ax) \ dx = x\ln(ax) - x + C$, which makes this integral easy.

Step 1: The formula for Integration By Parts is: $\int udv = uv - \int vdu$

The trick is that we have to let $dv = dx$.

$$\text{Let } u = \ln 2x \qquad \text{and } dv = dx$$

$$du = \frac{2}{2x}dx = \frac{1}{x}dx \text{ and } v = x$$

Plugging in to the formula we get:

$$\int \ln 2x dx = x\ln 2x - \int dx = x\ln 2x - x + C.$$

The answer is (D).

Problem 42. For the function $f(x) = \begin{cases} ax^3 - 6x; \text{if } x \le 1 \\ bx^2 + 4; x > 1 \end{cases}$ to be continuous and differentiable, a must be

This question is testing your knowledge of the rules of Continuity, where we also discuss differentiability.

Step 1: If the function is continuous, then if we plug 1 into the top and bottom pieces of the function we should get the same answer.

$$a(1^3) - 6(1) = b(1^2) + 4$$
$$a - 6 = b + 4$$

Step 2: If the function is differentiable, then if we plug 1 into the derivatives of the top and bottom pieces of the function we should get the same answer.

$$3a(1^2) - 6 = 2b(1)$$
$$3a - 6 = 2b$$

Step 3: Now we have a pair of simultaneous equations. If we solve them, we get $a = -14$

The answer is (C).

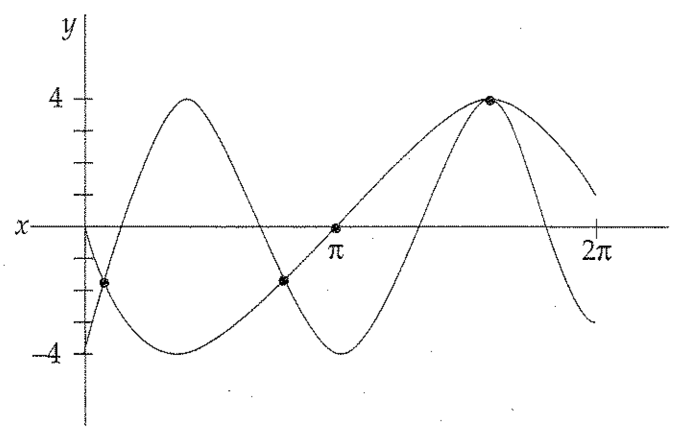
PROBLEM 43. Two particles leave the origin at the same time and move along the y-axis with their respective positions determined by the functions $y_1 = \cos 2t$ and $y_2 = 4 \sin t$ for $0 < t < 6$. For how many values of t do the particles have the same acceleration?

If you want to find acceleration, all you have to do is take the second derivative of the position functions.

Step 1: $\dfrac{dy_1}{dx} = -2\sin 2t$ and $\dfrac{dy_2}{dx} = 4\cos t$

$$\frac{d^2y_1}{dx^2} = -4\cos 2t \quad \text{and} \quad \frac{d^2y_2}{dx^2} = -4\sin t$$

Step 2: Now all we have to do is to graph both of these equations on the same set of axes on a calculator. You should make the window from $x = 0$ to $x = 7$ (leave yourself a little room so that you can see the whole range that you need). You should get a picture that looks like this:



Where the graphs intersect, the acceleration is the same. There are three points of intersection.

The answer is (D).

Problem 44. Find the distance traveled (to three decimal places) in the first four seconds, for a particle whose velocity is given by $v(t)=7e^{-t^2}$; where t stands for time.

Step 1: If we want to find the distance traveled, we take the integral of velocity from the starting time to the finishing time. Therefore, we need to evaluate $\int_0^4 7e^{-t^2}\,dt$.

Step 2: But we have a problem! We can't take the integral of e^{-t^2}. This means that the AP wants you to find the answer using your calculator.

Rounded to three decimal places, the answer is 6.204.

The answer is (B).

Problem 45. $\int \tan^6 x \sec^2 x\,dx =$

We can do this integral with u-substitution.

Step 1: Let $u=\tan x$. Then $du=\sec^2 x\,dx$.

Step 2: Substituting, we get: $\int \tan^6 x \sec^2 x\,dx = \int u^6 du$.

Step 3: This is an easy integral: $\int u^6 du = \frac{u^7}{7}+C$.

Step 4: Substituting back, we get: $\frac{\tan^7 x}{7}+C$.

The answer is (A).

ANSWERS AND EXPLANATIONS TO SECTION II

Problem 1. Consider the equation $x^2 - 2xy + 4y^2 = 64$.

(a) Write an expression for the slope of the curve at any point (x, y).

Step 1: The slope of the curve is just the derivative. But, here, we have to use Implicit Differentiation to find the derivative. If we take the derivative of each term with respect to x, we get:

$$2x\frac{dx}{dx} - 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 8y\frac{dy}{dx} = 0$$

Remember that $\frac{dx}{dx} = 1$, which gives us:

$$2x - 2\left(x\frac{dy}{dx} + y\right) + 8y\frac{dy}{dx} = 0$$

Step 2: Now just simplify and solve for $\frac{dy}{dx}$.

$$2x - 2x\frac{dy}{dx} - 2y + 8y\frac{dy}{dx} = 0$$

$$x - x\frac{dy}{dx} - y + 4y\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 4y\frac{dy}{dx} = y - x$$

$$(4y - x)\frac{dy}{dx} = y - x$$

$$\frac{dy}{dx} = \frac{y - x}{4y - x}$$

(b) Find the equation of the tangent lines to the curve at the point $x = 2$.

Step 1: We are going to use the point-slope form of a line, $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the curve and the derivative at that point is the slope m. First, we need to know the value of y when $x = 2$. If we plug 2 for x into the original equation, we get:

$$4 - 4y + 4y^2 = 64$$

$$4y^2 - 4y - 60 = 0$$

Using the quadratic formula, we get:

$$y = \frac{1 \pm \sqrt{61}}{2} = 4.41, -3.41$$

Notice that there are two values of y when $x = 2$, which is why there are two tangent lines.

Step 2: Now that we have our points, we need the slope of the tangent line at $x = 2$.

$$\frac{dy}{dx} = \frac{y - x}{4y - x}.$$

$$\text{At } y = 4.41, \frac{dy}{dx} = \frac{4.41 - 2}{4(4.41) - 2} = 0.15$$

$$\text{At } y = -3.41, \frac{dy}{dx} = \frac{-3.41 - 2}{4(-3.41) - 2} = 0.35$$

Step 3: Plugging into our equation for the tangent line, we get:

$$y - 4.41 = 0.15(x - 2)$$
$$y + 3.41 = 0.35(x - 2)$$

It is not necessary to simplify these equations.

(c) Find $\frac{d^2y}{dx^2}$ at $(0,4)$.

Step 1: Once we have the first derivative, we have to differentiate again to find $\frac{d^2y}{dx^2}$.

But, we have to use implicit differentiation again.

$$\frac{dy}{dx} = \frac{y - x}{4y - x}$$

Using the quotient rule $\frac{d^2y}{dx^2} = \dfrac{(4y - x)\left(\frac{dy}{dx} - \frac{dx}{dx}\right) - (y - x)\left(4\frac{dy}{dx} - \frac{dx}{dx}\right)}{(4y - x)^2}$

Simplifying we get: $\dfrac{d^2y}{dx^2} = \dfrac{(4y-x)\left(\dfrac{dy}{dx}-1\right)-(y-x)\left(4\dfrac{dy}{dx}-1\right)}{(4y-x)^2}$

Now, we Plug In $\dfrac{y-x}{4y-x}$ for $\dfrac{dy}{dx}$, which gives us:

$$\frac{d^2y}{dx^2} = \frac{(4y-x)\left(\dfrac{y-x}{4y-x}-1\right)-(y-x)\left(4\dfrac{y-x}{4y-x}-1\right)}{(4y-x)^2}$$

Now we would have to use a lot of algebra to simplify this but, fortunately, we can just plug (0, 4) in immediately for x and y, and solve from there.

$$\frac{d^2y}{dx^2} = \frac{(16)\left(\dfrac{4}{16}-1\right)-(4)\left(4\dfrac{4}{16}-1\right)}{(16)^2} = \frac{-3}{64}$$

Problem 2. A particle moves along the x-axis so that its acceleration at any time $t > 0$ is given by $a(t) = 12t - 18$. At time $t = 1$, the velocity of the particle is $v(1) = 0$ and the position is $x(1) = 9$.

(a) Write an expression for the velocity of the particle $v(t)$.

Step 1: We know that the derivative of velocity with respect to time is acceleration, so the integral of acceleration with respect to time is velocity.

$$\int a(t)\,dt = v(t).\ \int 12t - 18\,dt = 6t^2 - 18t + C = v(t)$$

If we Plug In the information that at time $t = 1$, $v(1) = 0$, we can solve for C.

$$6(1)^2 - 18(1) + C = 0$$
$$-12 + C = 0$$
$$C = 12$$

This means that the velocity of the particle is $6t^2 - 18t + 12$.

(b) At what values of t does the particle change direction?

When a particle is in motion, it changes direction at the time when its velocity is zero. (As long as acceleration is not also zero.) So all we have to do is set velocity equal to zero and solve for t.

$$6t^2 - 18t + 12 = 0$$
$$t^2 - 3t + 2 = 0$$
$$(t-2)(t-1) = 0$$
$$t = 1, 2$$

(c) Write an expression for the position $x(t)$ of the particle.

We know that the derivative of position with respect to time is velocity, so the integral of velocity with respect to time is position. $\int v(t)dt = x(t)$.

$$\int 6t^2 - 18t + 12\,dt = 2t^3 - 9t^2 + 12t + C = x(t)$$

If we Plug In the information that at time $t = 1$, $x(1) = 9$,we can solve for C.

$$2(1)^3 - 9(1)^2 + 12(1) + C = 9$$
$$5 + C = 9$$
$$C = 4 \text{ so } x(t) = 2t^3 - 9t^2 + 12t + 4$$

(d) Find the total distance traveled by the particle from $t = \frac{3}{2}$ to $t = 6$?

Step 1: Normally, all that we have to do to find the distance traveled is to integrate the velocity equation from the starting time to the ending time. But we have to watch out for whether the particle changes direction. If so, we have to break the integration into two parts—a positive integral for when it is traveling to the right, and a negative integral for when it is traveling to the left.

We know that the particle changes direction at $t = 1$ and at $t = 2$. We need to know which direction the particle is moving at time $t = \frac{3}{2}$. We can do this by plugging $\frac{3}{2}$ into the velocity and looking at its sign. $6\left(\frac{3}{2}\right)^2 - 18\left(\frac{3}{2}\right) + 12 = -\frac{3}{2}$.

This is negative, so the particle is moving to the left from $t = \frac{3}{2}$ to $t = 2$. And, because we know that the particle changes direction at $t = 2$, it must be moving to the right after that. Therefore we are going to need two integrals, one for $t = \frac{3}{2}$ to $t = 2$ and one for $t = 2$ to $t = 6$. So we need to evaluate:

$$\int_{\frac{3}{2}}^{2} -\left(6t^2 - 18t + 12\right)dt + \int_{2}^{6}\left(6t^2 - 18t + 12\right)dt$$

We already integrated these in part (c), $\int(6t^2 - 18t + 12)dt = 2t^3 - 9t^2 + 12t + C$, but now, instead of solving for the constant, we evaluate the equation at the limits of integration:

$$-\left(2t^3 - 9t^2 + 12t\right)\Big|_{\frac{3}{2}}^{2} + \left(2t^3 - 9t^2 + 12t\right)\Big|_{2}^{6}$$

$$= -\left(2(2)^3 - 9(2)^2 + 12(2)\right) + \left(2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + 12\left(\frac{3}{2}\right)\right) +$$
$$\left(2(6)^3 - 9(6)^2 + 12(6)\right) - \left(2(2)^3 - 9(2)^2 + 12(2)\right)$$

$$= -4 + 4.5 + 180 - 4 = 176.5 \text{ or } \frac{353}{2}$$

Problem 3. Let R be the region enclosed by the graphs of $y = 2 \ln x$ and $y = \frac{x}{2}$, and the lines $x = 2$ and $x = 8$.

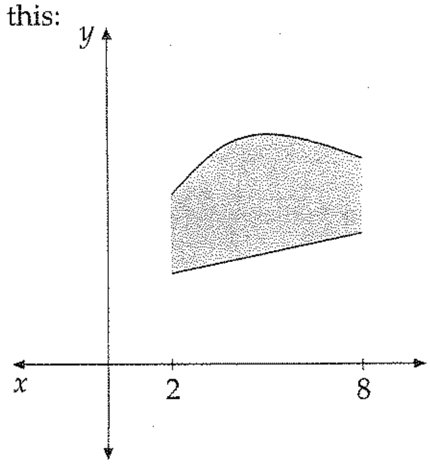
(a) Find the area of R.

Step 1: If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a, b]$, then the area of the region between the two curves is found by:

$$\int_a^b (f(x) - g(x))dx$$

In order to determine whether one of the curves is above the other, we can graph them on the calculator.

The graph looks like this:



As we can see, the graph of $y = 2 \ln x$ is above $y = \frac{x}{2}$ on the entire interval, so all we have to do is evaluate the integral $\int_2^8 \left(2\ln x - \frac{x}{2}\right)dx =$

Step 2: We can do the integration one of two ways—on the calculator or analytically.

Calculator: Evaluate **fnint** $\left(\left(2\ln x-\left(\frac{x}{2}\right)\right),x,2,8\right)=3.498$

Analytically: $\int_2^8\left(2\ln x-\frac{x}{2}\right)dx=2\int_2^8\ln x\,dx-\frac{1}{2}\int_2^8 x\,dx=$

$$=2(x\ln x-x)\Big|_2^8-\frac{1}{2}\left(\frac{x^2}{2}\right)\Bigg|_2^8=18.498-15=3.498$$

By the way, you should have memorized $\int\ln x\,dx=x\ln x-x$, or you can do it as one of the basic Integration By Parts integrals.

(b) Set up, but *do not integrate*, an integral expression, in terms of a single variable, for the volume of the solid generated when R is revolved about the *x-axis*.

Step 1: If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a, b]$, then the volume of the solid generated when the region is revolved about the x-axis is found by using the method of washers:

$$\pi\int_a^b[f(x)]^2-[g(x)]^2\,dx$$

Here, we already know that $f(x)$ is above $g(x)$ on the interval, so the integral we need to evaluate is:

$$\pi\int_2^8[2\ln x]^2-\left[\frac{x}{2}\right]^2dx$$

(c) Set up, but <u>do not integrate</u>, an integral expression, in terms of a single variable, for the volume of the solid generated when R is revolved about the line $x=-1$.

Step 1: Now we have to revolve the area around a <u>vertical</u> axis. If there are two curves, $f(x)$ and $g(x)$, where $f(x)$ is always above $g(x)$, on the interval $[a, b]$, then the volume of the solid generated when the region is revolved about the y-axis is found by using the method of shells:

$$2\pi\int_a^b x[f(x)-g(x)]dx$$

When we are rotating around a vertical axis, we use the same formula as when we rotate around the y-axis, but we have to account for the shift away from $x = 0$. Here we have a curve that is 1 unit farther away from the line $x = -1$ than it is from the y-axis, so we add 1 to the radius of the shell (For a more detailed explanation of shifting axes, see the unit on Finding the Volume of a Solid of Revolution). This gives us the equation:

$$2\pi\int_{2}^{8}(x+1)\left[2\ln x-\frac{x}{2}\right]dx$$

Problem 4. Water is draining at the rate of $48\pi\, ft^3/\text{sec}$ from a conical tank whose diameter at its base is 40 feet and whose height is 60 feet.

(a) Find an expression for the volume of water (in ft^3/sec) in the tank in terms of its radius.

The formula for the volume of a cone is: $V=\frac{1}{3}\pi R^2 H$, where R is the radius of the cone, and H is the height. The ratio of the height of a cone to its radius is constant at any point on the edge of the cone, so we also know that $\frac{H}{R}=\frac{60}{20}=3$. (Remember that the radius is half the diameter.) If we solve this for H and substitute, we get:

$$H = 3R$$
$$V=\frac{1}{3}\pi R^2(3R)=\pi R^3$$

(b) At what rate (in ft/sec) is the radius of the water in the tank shrinking when the radius is 16 feet?

Step 1: This is a Related Rates question. We now have a formula for the volume of the cone in terms of its radius, so if we differentiate it in terms of t we should be able to solve for the rate of change of the radius $\frac{dR}{dt}$.

We are given that the rate of change of the volume and the radius are, respectively:

$$\frac{dV}{dt}=48\pi \text{ and } R=16$$

Differentiating the formula for the volume, we get: $\frac{dV}{dt}=3\pi R^2\frac{dR}{dt}$.

Now we Plug In and get: $48\pi = 3\pi 16^2\frac{dR}{dt}$. Finally, if we solve for $\frac{dR}{dt}$, we get:

$$\frac{dR}{dt}=\frac{1}{16}\, ft/\text{sec}$$

(c) How fast (in ft/sec) is the height of the water in the tank dropping at the instant that the radius is 16 feet?

Step 1: This is the same idea as the previous problem, except that we want to solve for $\frac{dH}{dt}$. In order to do this, we need to go back to our ratio of height to radius and solve it for the radius:

$$\frac{H}{R} = 3 \quad \text{or} \quad \frac{H}{3} = R.$$

Substituting for R in the original equation, we get: $V = \frac{1}{3}\pi\left(\frac{H}{3}\right)^2 H = \frac{\pi H^3}{27}$.

Step 2: Now we need to know what H is when R is 16. Using our ratio:

$$H = 3\,(16) = 48.$$

Step 3: Now if we differentiate we get:

$$\frac{dV}{dt} = \frac{\pi H^2}{9}\frac{dH}{dt}$$

Now we Plug In and solve:

$$48\pi = \frac{\pi(48)^2}{9}\frac{dH}{dt}$$

$$\frac{dH}{dt} = \frac{3}{16}$$

One should also note that, because $H = 3R$, $\frac{dH}{dt} = 3\frac{dR}{dt}$. Thus, after we found $\frac{dR}{dt}$ in part 2, we merely had to multiply it by 3 to find the answer for part 3.

Problem 5. Let f be the function given by $y = f(x) = 2x^4 - 4x^2 + 1$.

(a) Find an equation of the line tangent to the graph at $(-2, 17)$.

In order to find the equation of a Tangent Line at a particular point we need to take the derivative of the function and Plug In the x and y values at that point to give us the slope of the line.

Step 1: The derivative is: $f'(x) = 8x^3 - 8x$. If we Plug In $x = -2$, we get:

$$f'(-2) = 8(-2)^3 - 8(2) = -48$$

This is the slope m.

Step 2: Now we use the slope-intercept form of the equation of a line, $y - y_1 = m(x - x_1)$, and Plug In the appropriate values of x, y, and m.

$$y - 17 = -48(x + 2)$$

If we simplify this we get $y = -48x - 79$.

(b) Find the x- and y-coordinates of the relative maxima and relative minima.

If we want to find the maxima/minima, we need to take the derivative and set it equal to zero. The values that we get are called critical points. We will then test each point to see if it is a maximum or a minimum.

Step 1: We already have the first derivative from part (a), so we can just set it equal to zero:

$$8x^3 - 8x = 0$$

If we now solve this for x we get:

$$8x(x^2 - 1) = 0 \quad 8x(x + 1)(x - 1) = 0 \quad x = 0, 1, -1.$$

These are our critical points. In order to test if a point is a maximum or a minimum, we usually use the *second derivative test*. We plug each of the critical points into the second derivative. If we get a positive value, the point is a relative minimum. If we get a negative value, the point is a relative maximum. If we get zero, the point is a point of inflection.

Step 2: The second derivative is $f''(x) = 24x^2 - 8$. If we Plug In the critical points we get:

$$f''(0) = 24(0)^2 - 8 = -8$$
$$f''(1) = 24(1)^2 - 8 = 16$$
$$f''(-1) = 24(-1)^2 - 8 = 16$$

So $x = 0$ is a relative maximum, and $x = 1, -1$ are relative minima.

Step 3: In order to find the y-coordinates, we plug the x values back into the original equation, and solve.

$f(0) = 1$

$f(1) = -1$

$f(-1) = -1$

and our points are

$(0,1)$ is a relative maximum

$(1,-1)$ is a relative minimum

$(-1,-1)$ is a relative minimum

(c) Find the x-coordinates of the points of inflection.

If we want to find the points of inflection, we set the second derivative equal to zero. The values that we get are the x-coordinates of the points of inflection.

Step 1: We already have the second derivative from part (b), so all we have to do is set it equal to zero and solve for x:

$$24x^2 - 8 = 0 \quad x^2 = \frac{1}{3} \quad x = \pm\sqrt{\frac{1}{3}}.$$

Step 2: In order to find the y-coordinates, we plug the x values back into the original equation, and solve.

$$f\left(\sqrt{\frac{1}{3}}\right) = 2\left(\sqrt{\frac{1}{3}}\right)^4 - 4\left(\sqrt{\frac{1}{3}}\right)^2 + 1 = \frac{2}{9} - \frac{4}{3} + 1 = -\frac{1}{9}$$

$$f\left(-\sqrt{\frac{1}{3}}\right) = 2\left(-\sqrt{\frac{1}{3}}\right)^4 - 4\left(-\sqrt{\frac{1}{3}}\right)^2 + 1 = \frac{2}{9} - \frac{4}{3} + 1 = -\frac{1}{9}$$

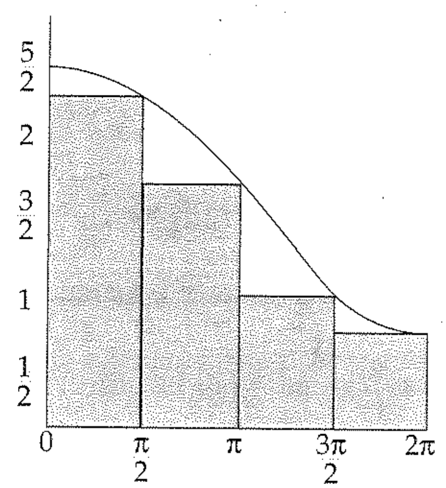
So the points of inflection are: $\left(\sqrt{\frac{1}{3}}, -\frac{1}{9}\right)$ and $\left(-\sqrt{\frac{1}{3}}, -\frac{1}{9}\right)$.

Problem 6. Let $F(x) = \int_0^x \left[\cos\left(\frac{t}{2}\right) + \left(\frac{3}{2}\right)\right] dt$ on the closed interval $[0, 4\pi]$.

(a) Approximate $F(2\pi)$ using four inscribed rectangles.

Step 1: This means that we need to find $\int_0^{2\pi} \left[\cos\left(\frac{t}{2}\right) + \left(\frac{3}{2}\right)\right] dt$.

The graph of $\cos\left(\frac{t}{2}\right) + \left(\frac{3}{2}\right)$ from 0 to 2π, using four inscribed rectangles looks like:



If we are cutting the interval $[0,2\pi]$ into 4 rectangles, the width of each rectangle is $\frac{\pi}{2}$.

The height of each rectangle depends on the x-coordinate.

Step 2: We can now set up the calculation for the area of the rectangles:

$$\text{Area} = \frac{\pi}{2}\left[\left(\cos\frac{\pi}{4}+\frac{3}{2}\right)+\left(\cos\frac{\pi}{2}+\frac{3}{2}\right)+\left(\cos\frac{3\pi}{4}+\frac{3}{2}\right)+\left(\cos\pi+\frac{3}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)+\left(\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{2}\right)\right]=\frac{5\pi}{2}\approx 7.854$$

(b) Find $F'(2\pi)$

Step 1: The Second Fundamental Theorem of Calculus says that if $f(x)$ is a continuous function, and a is a constant, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

So here we have: $\frac{d}{dx}\int_0^x \cos\left(\frac{t}{2}\right)+\frac{3}{2}\,dt = \cos\left(\frac{x}{2}\right)+\frac{3}{2}$

Step 2: Now we Plug In 2π for x and we get $\cos\pi+\frac{3}{2}=\frac{1}{2}$.

This was worth 2 points—1 for the correct derivative and 1 for the correct answer.

(c) Find the average value of $F'(x)$ on the interval $[0, 4\pi]$.

Step 1: The Mean Value Theorem for Integrals says that if you want to find the average value of $f(x)$ on an interval $[a, b]$, you need to evaluate the integral $\frac{1}{b-a}\int_a^b f(x)dx$. So here we would evaluate the integral $\frac{1}{4\pi-0}\int_0^{4\pi}\left(\cos\left(\frac{x}{2}\right)+\frac{3}{2}\right)dx$.

$$\frac{1}{4\pi}\int_0^{4\pi}\left(\cos\left(\frac{x}{2}\right)+\frac{3}{2}\right)dx = \frac{1}{4\pi}\int_0^{4\pi}\cos\frac{x}{2}\,dx+\frac{3}{8\pi}\int_0^{4\pi}dx =$$

$$\frac{1}{4\pi}2\sin\frac{x}{2}\Big|_0^{4\pi}+\frac{3}{8\pi}x\Big|_0^{4\pi} = \frac{1}{2\pi}\sin\frac{x}{2}\Big|_0^{4\pi}+\frac{3}{8\pi}x\Big|_0^{4\pi}$$

Step 2: Now we evaluate at the limits of integration and we get:

$$\frac{1}{2\pi}\sin\frac{x}{2}\Big|_0^{4\pi}+\frac{3}{8\pi}x\Big|_0^{4\pi} = \frac{1}{2\pi}(\sin 2\pi-\sin 0)+\frac{3}{8\pi}(4\pi)=\frac{3}{2}$$